Learning-Guided Higher-Order Automated Reasoning for Isabelle/HOL

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Outline

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- Two complementary learning problems:
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  - **Clause guidance**: which clauses to explore inside the ATP?
- Both problems have been successfully tackled in the first-order (FO) setting.

Learning-Guided Clause Selection

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- This work: extend both to the higher-order (HO) setting using Isabelle/HOL exports.

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  - TH0: monomorphic higher-order. $[(i \to i) \to \mathsf{list}_i \to \mathsf{list}_i]$
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- MeSh premise filter; 512 facts per slice; no ITE/let/choice.
- Total: 246,277 problems per dialect.

Bugs Found and Fixed

Running the export at scale revealed several issues:

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- HOL-Nominal `Class1`: clashes between names and conames.
- Smaller monomorphisation issues in lower-logic encodings.
- All fixed in the Isabelle development version.
- These fixes benefit *all* provers, not just ours.

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Higher-Order Saturation Provers

Both E and Vampire use a given-clause saturation loop:

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- Clause selection is the key decision point — target for learning.

λENIGMA: HO Features for ENIGMA

- In λE, HO applications (F X Y) are stored as @(F, X, Y) using an internal symbol @.
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  - @(F, X, Y): X, Y treated as children of F, not siblings: F(X,Y).
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- Works on TF0 and TH0.

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- Adapted to handle polymorphic HO problems:
  - New GNN node type for type constructors.
  - Sorts promoted to full expressions (with sort variables).
  - Distinct node labels for universal syntactic concepts: arrow type, apply, λ, de Bruijn indices, propositional connectives.

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- Built on VAMPIRE's `hol` branch.
- Works on TF0, TH0, and TH1.

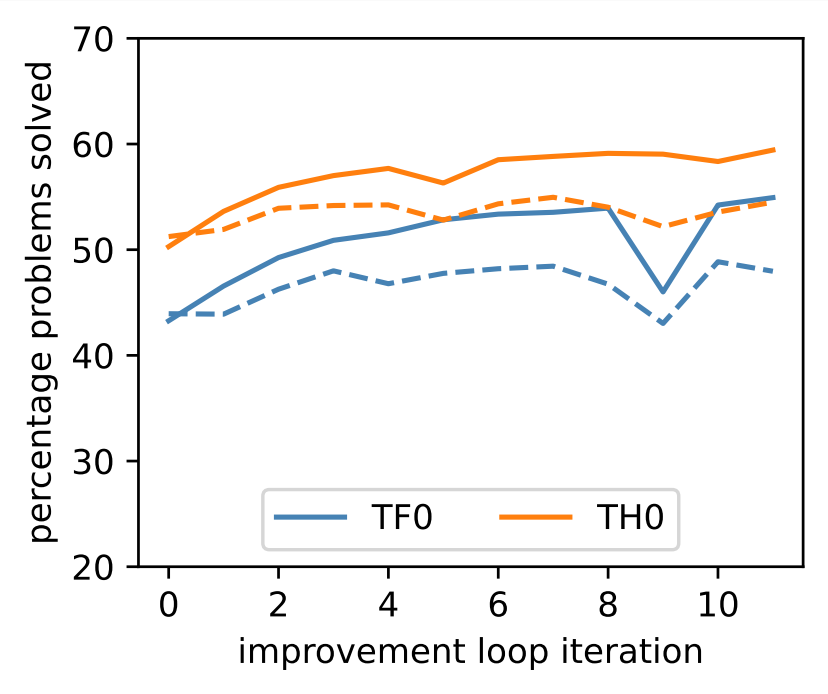
Outline

Experimental Setup

- 30,000 training problems; 5,000 test problems (aligned across dialects).
- λ**ENIGMA**:
  - 10 training iterations; $\sim$12 h per iteration.
  - 10 s per problem.
  - Evaluated on TF0 and TH0.
- **Deepire-2.0-HO**:
  - 10 training iterations; $\sim$4.5 days per dialect.
  - 16 billion CPU instructions ($\sim$6 s).
  - Evaluated on TF0, TH0, and TH1.

Training Plots: λENIGMA and Deepire-2.0-HO

left = λENIGMA



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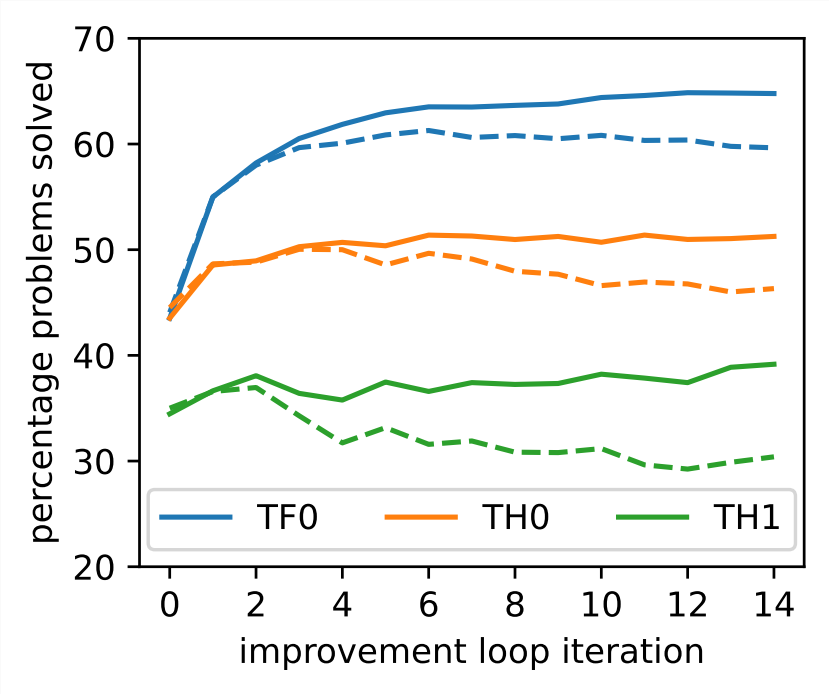

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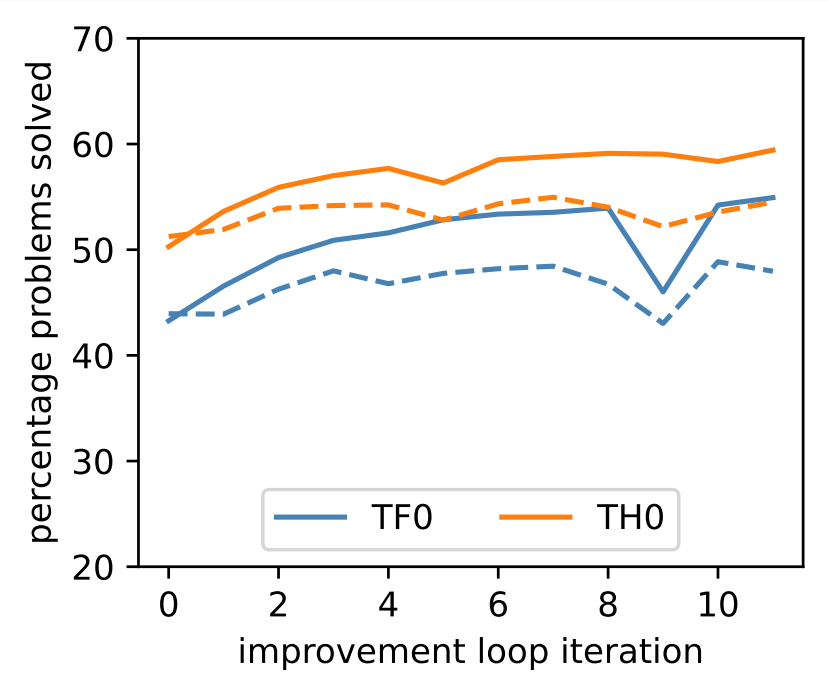

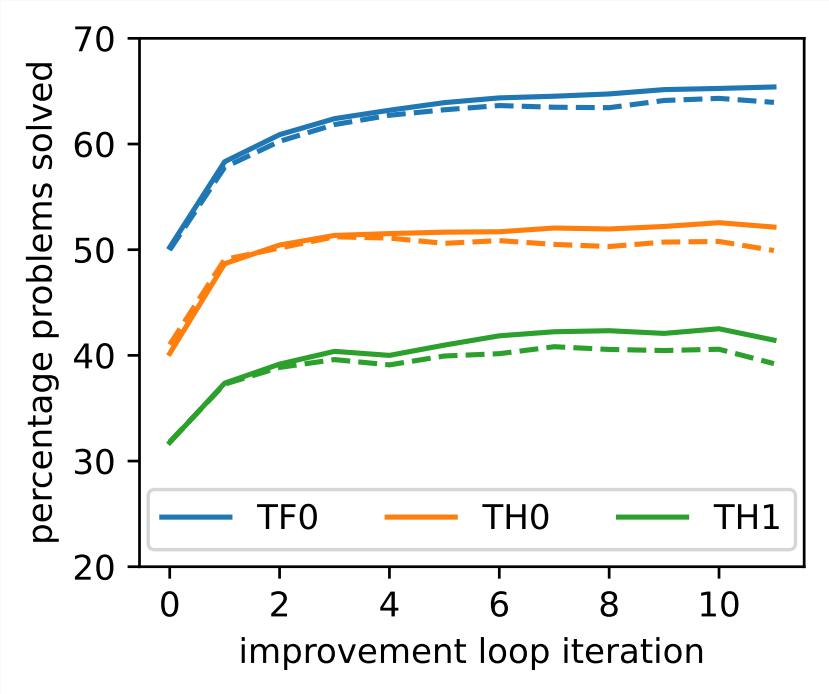

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Results: λENIGMA

- λE baseline on TH0 outperforms TF0 — and even VAMPIRE.
- λENIGMA improves consistently on both fragments:
  - +7.26 % on TH0 (test).
  - +11.2 % on TF0 (test).
- Occasional dips (e.g. iteration 5 for TF0) recover; train and test curves remain correlated.

Results: Deepire-2.0-HO

- Vampire + Deepire 2.0 on TF0 is still the strongest single dialect: ∼65 % test coverage.
- Deepire-2.0-HO brings comparable relative gains on HO fragments:
  - TH0: 40.2 % → 51.2 % (+27 % relative).
  - TH1: 31.8 % → 40.8 % (+28 % relative).
- HO starting points are lower than FO — understanding why is an open research question.

Complementarity of Strategies

All five best strategies (2 FO + 3 HO) evaluated on the 5,000 test problems:

System	*Fragment*	*Solves*	*Total*	*Cumulative*
Deepire 2.0	TF0	3064	3064	61.28 %
λENIGMA	TH0	2748	3329	66.58 %
Deepire-2.0-HO	TH1	1848	3377	67.54 %
Deepire-2.0-HO	TH0	2502	3404	68.08 %
λENIGMA	TF0	2443	**3425**	**68.50** %

- 3 HO strategies alone: 60.92 %
- 2 FO strategies alone: 62.58 %
- All five together: 68.50 %.
- HO strategies add +5.92 % over FO alone.

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Summary

Contributions:

- λENIGMA: first learning-guided HO clause guidance for λE (TH0 + TF0 from Isabelle).
- Deepire-2.0-HO: neural HO clause guidance for Vampire (TH0 + TH1).
- Improved Sledgehammer export: fixed bugs in HOL-Analysis and HOL-Nominal; improved monomorphisation.
- Consistent improvement over baselines; HO strategies are complementary to FO ones.

Future Work

- Why does native TH1 solving currently lag behind TH0?
  $\Rightarrow$ Investigate calculus, encoding, and strategy differences.
- Use HO premise selection to improve TH1 training data quality.
- Scale training to the full 246,277 problems
  (currently too slow; GPU training would help).
- Closer integration between λENIGMA and Deepire-2.0-HO strategies.

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Thank you!